Name:______________________________________ **Date:**__________________

PSTAT 5A: MT1 Practice Problems

Summer Session A 2023, with *Ethan P. Marzban*

Please also take a look at the past exams posted to the GitHub site, for additional practice problems.

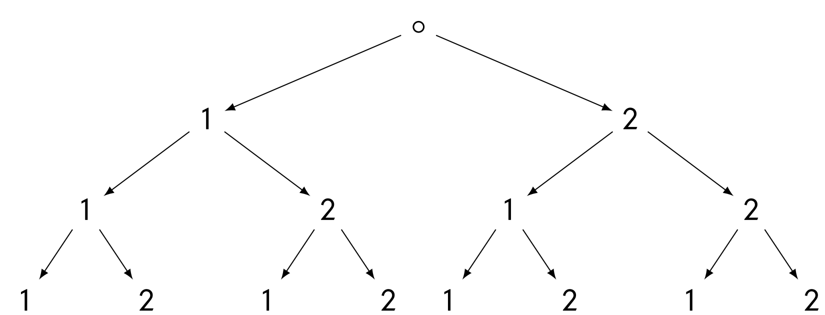
1. Three numbers are to be selected from the set $\{1, 2\}$. Suppose that the selection of numbers is done at random, and that the numbers are replaced after each selection (so the same number could be selected more than once). Additionally, suppose that the order in which the numbers are selected <u>is</u> important. The three numbers are then recorded.

   (a) Is it possible to express the outcome space Ω for this experiment as a table? If so, express Ω as a table. If not, explain why not.

   > **Solution:** It is not possible to express Ω as a table, since there are more than two stages to the experiment.

   (b) Express the outcome space Ω for this experiment as a tree.

   > **Solution:**



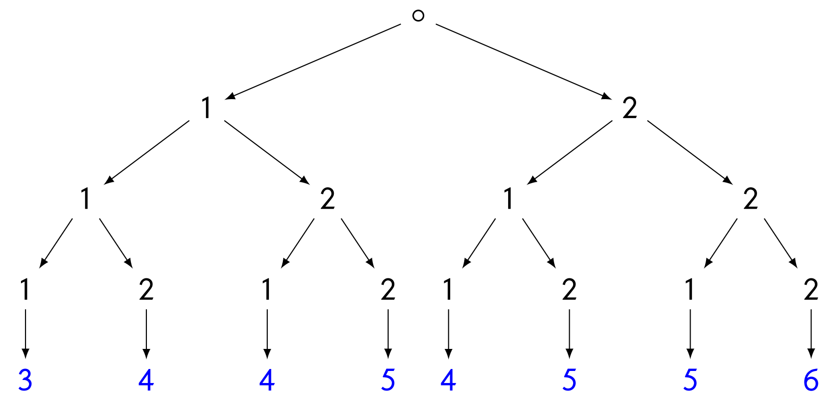
   (c) How many outcomes are in Ω? Justify your answer.

   > **Solution:** We can find the number of elements in Ω by counting the leaves on the tree in part (b), or by using counting arguments. In either case, we find the number of elements in Ω to be $\boxed{8}$.

   (d) Let F denote the event "the sum of the three numbers is equal to 4." Use the Classical Approach to Probability to compute $\mathbb{P}(F)$. A fully correct answer should list out the elements in F explicitly.

Name: ____________________ **Date:** ____________

Solution: Since we are dealing with the sum of the three numbers now, let us add an additional row to our table to track the sum corresponding to each outcome:



Now we see that the only outcomes that correspond to a sum of 4, and consequently the only outcomes that comprise the event F, are

$$F = \{(1,\ 1,\ 2),\ (1,\ 2,\ 1),\ (2,\ 1,\ 1)\}$$

which means $\#(F) = 3$ and so

$$\mathbb{P}(F) = \frac{\#(F)}{\#(\Omega)} = \boxed{\frac{3}{8}}$$

2. Suppose A and B are independent events with $\mathbb{P}(A) = 0.4$ and $\mathbb{P}(B) = 0.5$.

(a) Compute $\mathbb{P}(A \cup B)$.

Solution: By the **Addition Rule**,

$$\begin{aligned} \mathbb{P}(A \cup B) &= \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) \\ &= (0.4) + (0.5) - (0.4) \cdot (0.5) = \boxed{0.7} \end{aligned}$$

where we have utilized the fact that $A \perp B$ to conclude $\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$.

(b) Compute $\mathbb{P}(A^C \cap B^C)$.

Solution: By DeMorgan's Laws,

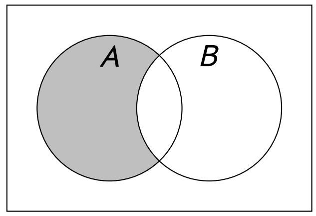
$$(A^C \cap B^C) = (A \cup B)^C$$

Hence, by the **Complement Rule**,

$$\mathbb{P}(A^\complement \cap B^\complement) = 1 - \mathbb{P}(A \cup B) = 1 - 0.7 = 0.3$$

(c) Compute $\mathbb{P}(A \cap B^\complement)$. **Hint:** Sketch a Venn Diagram.

Solution: Following the hint, we sketch a Venn Diagram:



We see, therefore, that

$$\begin{aligned} \mathbb{P}(A \cap B^\complement) &= \mathbb{P}(A) - \mathbb{P}(A \cap B) \\ &= (0.4) - (0.5) \cdot (0.4) = 0.2 \end{aligned}$$

3. In a particular group of people, 50% have blue eyes and 20% wear glasses. Furthermore, 20% of those who have blue eyes wear glasses.

(a) What is the probability that a randomly-selected person from this group has blue eyes and wears glasses? Remember that a fully correct answer should clearly define events and notation, and translate the information provided in the problem statement into the notation established.

Solution: Let B denote the event "a randomly-selected person has blue eyes" and G denote the event "a randomly-selected person wears glasses". Then, from the information provided in the problem statement, we have

$$\mathbb{P}(B) = 0.5; \quad \mathbb{P}(G) = 0.2; \quad \mathbb{P}(G \mid B) = 0.2$$

We seek $\mathbb{P}(B \cap G)$, which can be computed using the **Multiplication Rule:**

$$\mathbb{P}(B \cap G) = \mathbb{P}(G \mid B) \cdot \mathbb{P}(B) = (0.2) \cdot (0.5) = 0.1$$

(b) What is the probability that a randomly-selected person from this group has blue eyes or wears glasses, <u>but not both</u>?

Name:________________________________ **Date:**________________

Solution: Using a result derived on Quiz 1,

$$\begin{aligned}\mathbb{P}[(B \cap G^C) \cup (B^C \cap G)] &= \mathbb{P}(B) + \mathbb{P}(G) - 2\mathbb{P}(B \cap G) \\ &= (0.5) + (0.2) - (0.2) = 0.5\end{aligned}$$

(c) Are the events "randomly-selected person has blue eyes" and "randomly-selected person wears glasses" independent? Why or why not?

Solution: Perhaps the easiest way to answer this question is to note

$$\mathbb{P}(G \mid B) = 0.2 = \mathbb{P}(G)$$

Hence, the events are independent. We could have also used the fact that

$$\mathbb{P}(B \cap G) = 0.1 = (0.5) \cdot (0.2) = \mathbb{P}(B) \cdot \mathbb{P}(G)$$

4. Daria collects data on the political beliefs and relationship status of several individuals. Below is a contingency table summarizing her results:

	Relationship		
Beliefs	Single	Dating	Married
Liberal	30	20	20
Moderate	40	25	25
Conservative	25	30	30

(a) What is the probability that a randomly-selected person has conservative beliefs?

Solution: Let A denote the event "a randomly-selected person has conservative beliefs". By the Classical Approach to Probability,

$$\begin{aligned}\mathbb{P}(A) &= \frac{\#(A)}{\#(\Omega)} \\ &= \frac{\text{number of people with conservative beliefs}}{\text{total number of people}} \\ &= \frac{25 + 30 + 30}{30 + 20 + 20 + 40 + 25 + 25 + 25 + 30 + 30} = \frac{85}{245}\end{aligned}$$

(b) What is the probability that a randomly-selected person is single?

Name: ______________________________ **Date:** _______________

Solution: Let B denote the event "a randomly-selected person is single". By the Classical Approach to Probability,

$$\begin{aligned}\mathbb{P}(B) &= \frac{\#(B)}{\#(\Omega)} \\ &= \frac{\text{number of people who are single}}{\text{total number of people}} \\ &= \frac{30 + 40 + 25}{245} = \boxed{\frac{95}{245}}\end{aligned}$$

(c) What is the probability that a randomly-selected person has moderate beliefs and is married?

Solution: Let C denote the event "a randomly-selected person has moderate beliefs" and D denote the event "a randomly selected person is married." We seek $\mathbb{P}(C \cap D)$, which, by the Classical Approach to Probability, can be computed as

$$\begin{aligned}\mathbb{P}(C \cap D) &= \frac{\#(C \cap D)}{\#(\Omega)} \\ &= \frac{\text{number of people who are married and have moderate beliefs}}{\text{total number of people}} \\ &= \boxed{\frac{25}{245}}\end{aligned}$$

(d) A person is selected at random, and it is noted that they are dating What is the probability that they have liberal beliefs?

Solution: Let E denote the event "a randomly-selected person is dating" and F denote the event "a randomly selected person has liberal beliefs" We seek $\mathbb{P}(F \mid E)$, which, by the Classical Approach to Probability, can be computed as

$$\begin{aligned}\mathbb{P}(F \mid E) &= \frac{\#(F \cap E)}{\#(E)} \\ &= \frac{\text{number of people who are dating and have liberal beliefs}}{\text{number of people who are dating}} \\ &= \frac{20}{20 + 25 + 30} = \boxed{\frac{20}{75}}\end{aligned}$$

5. Let $X = \{-1, 0, 1, 2, 10\}$.

(a) Compute $\overline{x}$, the mean of X.

Solution:

$$\begin{aligned}\overline{x} &= \frac{1}{n}\sum_{i=1}^{n} x_i \\ &= \frac{1}{5}(-1+0+1+2+10) = \frac{12}{5}\end{aligned}$$

(b) Compute s_X, the standard deviation of X.

Solution:

$$\begin{aligned}s_X^2 &= \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})^2 \\ &= \frac{1}{5-1}\left[\left(-1-\frac{12}{5}\right)^2 + \left(0-\frac{12}{5}\right)^2 + \left(1-\frac{12}{5}\right)^2 \right. \\ &\qquad \left. + \left(2-\frac{12}{5}\right)^2 + \left(10-\frac{12}{5}\right)^2\right] \\ &= \frac{193}{10} \\ s_X &= \sqrt{s_X^2} = \sqrt{\frac{193}{10}} = \frac{\sqrt{1930}}{10} \approx 4.393\end{aligned}$$

(c) Compute the **mean absolute deviation** (MAD) of X, defined as

$$\mathrm{mad}(X) = \frac{1}{n}\sum_{i=1}^{n}|x_i - \overline{x}|$$

Solution:

$$\begin{aligned}\mathrm{mad}(X) &= \frac{1}{n}\sum_{i=1}^{n}|x_i - \overline{x}| \\ &= \frac{1}{5}\left[\left|-1-\frac{12}{5}\right| + \left|0-\frac{12}{5}\right| + \left|1-\frac{12}{5}\right| + \left|2-\frac{12}{5}\right| + \left|10-\frac{12}{5}\right|\right] \\ &= \frac{19}{5} = 3.8\end{aligned}$$

(d) Compute range(X), the range of X.

Solution:

$$\text{range}\{X\} = \max\{X\} - \min\{X\} = 10 - (-1) = 11$$

6. A box contains tickets numbered 1 through 10. John reaches in and selects two tickets at random, not replacing the first ticket.

(a) Suppose that the order in which John draws the tickets is important. How many elements are in Ω, the outcome space for this experiment?

Solution: We have previously seen in lecture that when order matters, the number of ways to draw k objects from a total of n objects is $(n)_k$. Hence, the final answer should be $(10)_2$. If you did not remember this result, we could have used a slot diagram with two slots (one for each of the numbers John selects):

$$\underline{\quad} \;\; \underline{\quad}$$

There are 10 tickets initially, so we put a 10 in our first slot. After picking a number, there are only 9 remaining (since John doesn't replace his first ticket) meaning we put a 9 in our second slot, and finally multiply across (invoking the Fundamental Principle of Counting):

$$\underline{10} \times \underline{9}$$

Either way, we find the answer to be $(10)_2 = 90$.

(b) Suppose that the order in which John draws the tickets is not important. How many elements are in Ω, the outcome space for this experiment?

Solution: We have previously seen in lecture that when order does not matter, the number of ways to draw k objects from a total of n objects is $\binom{n}{k}$. Hence, the final answer should be $\binom{10}{2}$.

If you did not remember this result, here is how we could proceed: start by assuming order *did* matter. Then, we would have $(10)_2$ possibilities. Now, to relax the assumption of ordering we divide by the number of ways to rearrange 2 tickets in a line, which is 2!, giving

$$\frac{(10)_2}{2!} = \frac{10!}{2! \cdot 8!} = \binom{10}{2} = 45$$

Name:________________________________ **Date:**________________

(c) Suppose again that the order in which John draws the tickets is important. Use the Classical Approach to Probability to compute the probability that both of the numbers John selects are even.

Solution: Let E denote the event "both numbers selected are even", so that we seek $\mathbb{P}(E)$. The Classical Approach to Probability takes

$$\mathbb{P}(E) = \frac{\#(E)}{\#(\Omega)}$$

By part (a), $\#(\Omega) = (10)_2$. Since we are using the Classical Approach to probability, all that remains is to find $\#(E)$; i.e. the number of ways in which John can draw two even numbers. To find this, we use a slot diagram:

$$\underline{\quad}\ \underline{\quad}$$

Here, on the first slot there are only 5 elements John can pick (i.e. the 5 even numbers that are in the set $\{1, 2, \cdots, 10\}$); after picking an even number on the first draw, there are $5 - 1 = 4$ tickets remaining meaning our slot diagram looks like

$$\underline{5} \times \underline{4}$$

Hence, $\#(E) = (5)_2$ and so our final answer is

$$\boxed{\mathbb{P}(E) = \frac{(5)_2}{(10)_2} = \frac{20}{90} = 22.\overline{2}\%}$$

(d) Suppose again that the order in which John draws the tickets is not important. Use the Classical Approach to Probability to compute the probability that both of the numbers John selects are even.

Solution: Again let E denote the event "both numbers selected are even", so that we seek $\mathbb{P}(E)$. The Classical Approach to Probability takes

$$\mathbb{P}(E) = \frac{\#(E)}{\#(\Omega)}$$

By part (b), $\#(\Omega) = \binom{10}{2}$. Since we are using the Classical Approach to probability, all that remains is to find $\#(E)$; i.e. the number of ways in which John can draw two even numbers when order does not matter. Here is another way to think about this: there are 5 even numbers, and John wants to select 2 of these disregarding the order of selection.

Name:______________________________________ **Date:**____________________

The number of ways to do this is $\binom{5}{2}$; hence,

$$\mathbb{P}(E) = \frac{\binom{5}{2}}{\binom{10}{2}} = \frac{10}{45} = 22.\overline{2}\%$$

Name:____ **Date:**____________________

Some Practice Multiple Choice Questions

Problem 7 - 10 refer to the following situation: Jim would like to write a function called `sign()` that takes in a single input `x` and returns "`positive`" if `x` is positive, "`zero`" if `x` is zero, and "`negative`" if `x` is negative. To that end, he has written the following skeleton code but is missing some important pieces (assume this is the **only** code in Jim's Jupyter Notebook, and that there are **no** other code cells before or after):

```
def sign(x):
    if x __Blank 1__ 0:
        return "negative"
    __Blank 2__ x __Blank 3__ 0:
        return "zero"
    else:
        return "positive"
```

7. What should go in Blank 1?

    A. <

    B. <=

    C. >

    D. >=

    E. None of the above.

8. What should go in Blank 2?

    A. `else`

    B. `else if`

    C. `elif`

    D. `e_if`

    E. None of the above.

Name:________________________________ **Date:**________________

9. What should go in Blank 3?

   A. `=`

   B. `==`

   C. `!=`

   D. `=!`

   E. None of the above.

10. What is missing from the body of Jim's function (specifically, this is something we mentioned in Lab that should *always* be included with a function)

    A. An output statement

    B. A return statement

    C. An exception statement

    D. A docstring

    E. None of the above.

. . . .

11. Given two events E and F with $\mathbb{P}(E) = \mathbb{P}(F) = 0.5$ and $\mathbb{P}(E \cap F) = 0$, which of the following must be true?

    A. E and F are both disjoint and independent

    B. E and F are disjoint but not independent

    C. E and F are not disjoint but independent

    D. E and F are neither disjoint nor independent

12. Which module contains the function `Table()` that was used in Lab 2?

    A. `numpy`

    B. `tables`

    C. `datascience`

    D. `ds`

    E. None of the above

Name:____________________________________ **Date:**__________________

13. Suppose the variable `x`, in a Jupyter Notebook, is assigned the value `[1, 2, 3, 4]`. What is the result of running `x[2]`?

    A. `1`

    B. `2`

    C. `3`

    D. `4`

    E. None of the above